

II. *On the Attraction of the Himalaya Mountains, and of the elevated regions beyond them, upon the Plumb-line in India.* By the Venerable JOHN HENRY PRATT, M.A., Archdeacon of Calcutta. Communicated by the Rev. J. CHALLIS, M.A., F.R.S. &c.

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1. IT is now well known that the attraction of the Himalaya Mountains, and of the elevated regions lying beyond them, has a sensible influence upon the plumb-line in North India. This circumstance has been brought to light during the progress of the great trigonometrical survey of that country. It has been found by triangulation that the difference of latitude between the two extreme stations of the northern division of the arc, that is, between Kalianpur and Kaliana, is  $5^{\circ} 23' 42'' \cdot 294$ , whereas astronomical observations show a difference of  $5^{\circ} 23' 37'' \cdot 058$ , which is  $5'' \cdot 236^*$  less than the former.

2. That the geodetic operations are not in fault appears from this; that two bases, about seven miles long, at the extremities of the arc having been measured with the utmost care, and also the length of the northern base having been computed from the measured length of the southern one, through a chain of triangles stretching along the whole arc, about 370 miles in extent, the difference between the measured and the computed lengths of the northern base was only  $0 \cdot 6$  of a foot, an error which would produce, even if wholly lying in the meridian, a difference of latitude no greater than  $0'' \cdot 006$ .

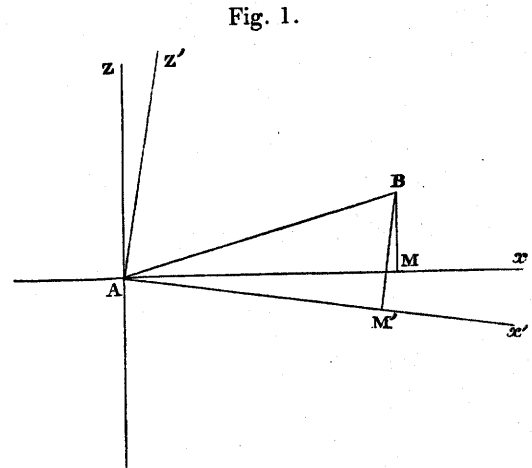
3. The difference  $5'' \cdot 236$  must therefore be attributed to some other cause than error in the geodetic operations. A very probable cause is the attraction of the superficial matter which lies in such abundance on the north of the Indian arc. This disturbing cause acts in the right direction; for the tendency of the mountain mass must be to draw the lead of the plumb-line at the northern extremity of the arc more to the north than at the southern extremity, which is further removed from the attracting mass. Hence the effect of the attraction will be to lessen the difference of latitude, which is the effect observed. Whether this cause will account for the error in the difference of latitude in *quantity*, as well as in direction, remains to be considered, and is the question I propose to discuss in the present paper.

4. But if mountain attraction have any sensible influence at the stations on the arc, how is it that the geodetic operations are not affected by it? How is it that such a remarkable degree of exactness between the measured and computed lengths of the

\* This is the difference as stated by Colonel EVEREST in his work on the Measurement of the Meridional Arc of India, published in 1847. See p. clxxviii.

northern base attests, it would seem, to the non-existence of any external disturbing cause? For in observing the altitude or depression of one station in the triangulation as seen from another, the error on the plumb-line must come into the calculation. The answer is, that these small errors occur in the calculation of the horizontal arc in very small terms not higher than the second order; whereas in the expression for the inclination of the two verticals at the extremities of the arc they occur in terms of the first order. This I will further illustrate.

5. Suppose the arc divided into  $n$  equal portions: and let  $\nu_0, \nu_2, \dots, \nu_n$  be the deflections of the plumb-line at the  $n+1$  stations thus chosen. Let A be one of these stations, and B the next towards the south; Az, Ax vertical and horizontal lines through A on the supposition that there is no mountain attraction; Az', Ax' the vertical and horizontal lines as affected by attraction. Draw BM and BM' perpendicular to Ax and Ax': let AM =  $a$ , BM =  $h$ ,  $\angle zAz' = \nu$ ,  $\angle BAM = \alpha$ . Then AM is the true horizontal distance between A and B, and AM' the calculated horizontal distance. Hence the calculation makes this portion of the arc too short by



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$$AM - AM' = AM \left( 1 - \frac{\cos(\alpha + \nu)}{\cos \alpha} \right) = a(\tan \alpha \cdot \sin \nu + 1 - \cos \nu) = h \cdot \nu + \frac{1}{2} a \cdot \nu^2,$$

neglecting the cube and higher powers of  $\nu$ .

Hence the whole arc is made too short by

$$h_0 \nu_0 + h_1 \nu_1 + h_2 \nu_2 + \dots + h_n \nu_n + \frac{a}{2} (\nu_0^2 + \nu_1^2 + \nu_2^2 + \dots + \nu_n^2),$$

$h_0, h_1, h_2, \dots, h_n$  being the heights of the various stations of observation above the true horizontal line. When the Station B is below A then  $h$  is negative. These heights are all extremely small compared with  $a$ , as the arc lies through a comparatively flat country. Hence the expression for the error in the length of the arc is made up, as I said, of small terms of no higher order than the second; whereas the error in the difference of latitude ( $= \nu_n - \nu_0$ ) has terms of the first order.

6. That this expression for the shortening of the arc is a minute quantity utterly inappreciable, may easily be shown by taking an extreme case. The quantities  $h_0, h_1, h_2, \dots, h_n$  are some of them positive and some of them negative, in such a manner that their algebraical sum equals the difference of height of Kalianpur and Kaliana above the level of the sea. From Colonel EVEREST'S work on the Indian Arc (published in 1847) I gather, that between Kalianpur and Kaliana there are forty-seven principal stations, or, including the two terminal ones, forty-nine: and the Survey

shows that in passing from north to south there are twenty-five elevations of one station above the level of the preceding one, amounting in all to 3901·1 feet; and twenty-three depressions, amounting to 2965·2 feet (see pp. 269–273). The difference between these = 935·9 feet, which is the height assigned to Kalianpur above Kaliana. I will take these, then, as the values of  $h_0h_1h_2\dots h_n$  in my present example; so that the sum of the positive quantities among  $h_0h_1h_2\dots = 3901·1$  feet, and the sum of the negative = 2965·2, and  $n=48$ . Now  $\nu_n$  is the greatest and  $\nu_0$  is the least of the quantities  $\nu_n\dots\nu_0$ . Hence it follows, that

$$h_0\nu_0 + h_1\nu_1 + \dots + h_n\nu_n \text{ is less than } 3901·1\nu_n - 2·965·2\nu_0,$$

and therefore, much more, less than  $3901·1\nu_n$  feet.

Now by the Survey  $\nu_n - \nu_0 = 5''·236$ , or in arcs =  $0·000025$ ;

$$\therefore h_0\nu_0 + h_1\nu_1 + \dots + h_n\nu_n \text{ is less than } 3901·1 \times 0·000025 \text{ or } 0·097527 \text{ foot.}$$

If in this extreme case of supposing the attraction to equal its greatest value at more than half of the stations, and that at stations where its effect would be greatest, the result is so insignificant, what must it be in the actual case\*? The same may be shown with respect to the other term in the expression for the shortening of the arc, viz.  $\frac{1}{2}a(\nu_0^2 + \nu_1^2 + \dots + \nu_n^2)$ . This quantity is less than  $\frac{n+1}{2} a \cdot \nu_n^2$ ; or, if we reckon the distance between Kaliana and Kalianpur to be 370 miles, and therefore  $n \cdot a = 370 \times 1760 \times 3$  feet and  $n=48$ , this quantity is less than 0·008 of a foot, which is utterly inappreciable. Hence mountain attraction may have a sensible value at the stations on the arc, and yet not affect the *geodetic* calculations in the slightest appreciable degree†.

7. I can see no ground, therefore, whatever for the process of dispersion which Colonel EVEREST describes at page clxx of the Introduction to his work, by which he distributes the error  $5''·236$  among the triangles. It appears to me to be unquestionable that the geodetic operations are in no way sensibly affected by mountain attraction, and therefore need no correction whatever on that account. It is the *astronomical* operation of observing the difference of latitude which requires the correction. That it is here that the correction must be applied appears again in attempting to determine the azimuths of the arc at seven stations *astronomically* (see p. xlii). It is only when the plumb-line is brought into use to determine the vertical angles of stars that the effect of attraction becomes sensible; and never in the geodetic calculations, where only horizontal angles or extremely minute vertical angles (viz. the elevations or depressions of  $h_0h_1h_2\dots$ ) are observed.

8. The importance of accounting satisfactorily for the difference between the geodetic

\* If the triangulation be carried into elevated regions some of the values of  $h_0h_1h_2\dots$  will be large; and therefore the conclusion in the text will not in that case stand.

† In this paper I show that the difference caused by attraction in the latitudes of the extremities of the northern division of the arc, viz. Kaliana and Kalianpur, amounts to  $15''·885$ , which is more than three times the angle  $5''·236$ . But the conclusion arrived at in art. 6. is still true.

and astronomical results appears from the effect it must have upon the determination of the earth's ellipticity; an effect such, that unless this quantity be fully accounted for, it must render the great Indian Survey comparatively useless in the delicate problem of the Figure of the Earth, however valuable it may be for the purposes of mapping the vast continent of Hindostan.

9. The effect of a small error in the difference of latitude upon the determination of the ellipticity may be calculated as follows:—

Let  $\varepsilon$  be the ellipticity, a quantity known not to differ much from  $\frac{1}{300}$ ;  $\lambda$  the amplitude of the arc;  $\mu$  the latitude of the middle point of the arc. Then by the usual formula

$$\frac{\text{length of arc}}{\text{equatorial radius}} = \lambda - \frac{1}{2}\varepsilon(\lambda + 3 \sin \lambda \cos 2\mu).$$

But  $\sin \lambda = \lambda - \frac{1}{6}\lambda^3 + \dots = \lambda \left(1 - \frac{1}{6}\lambda^2 + \dots\right)$ ;  $\lambda = 5^\circ 23' 37''$  for the arc between Kalianpur and Kalia = 0.094 in parts of the radius,

$$\therefore \frac{1}{6}\lambda^2 = 0.00147.$$

Hence by putting  $\lambda$  instead of  $\sin \lambda$  in the above formula, we shall be omitting a quantity of the order  $\frac{1}{2}\varepsilon \times 0.00441 \cos 2\mu$ , which is utterly insignificant,

$$\therefore \frac{\text{length of arc}}{\text{equatorial radius}} = \lambda \left(1 - \frac{1}{2}\varepsilon\right) \left(1 - \frac{3}{2}\varepsilon \cos 2\mu\right).$$

In the same way if  $L$  be the amplitude and  $M$  the latitude of the middle point of another arc,

$$\frac{\text{length of arc } L}{\text{equatorial radius}} = L \left(1 - \frac{1}{2}\varepsilon\right) \left(1 - \frac{3}{2}\varepsilon \cos 2M\right),$$

$$\therefore \frac{\text{length of arc } \lambda}{\text{length of arc } L} = \frac{\lambda}{L} \left\{1 - \frac{3}{2}\varepsilon(\cos 2\mu - \cos 2M)\right\}.$$

Suppose the observed values of  $\lambda$  and  $\mu$  are subject to small errors owing to mountain attraction; to find the effect on  $\varepsilon$  we must differentiate this expression, supposing the angles  $\lambda$ ,  $\mu$  and  $\varepsilon$  variable and the other quantities constant,

$$\therefore 0 = d\lambda \left\{1 - \frac{3}{2}\varepsilon(\cos 2\mu - \cos 2M)\right\}$$

$$+ 3\lambda\varepsilon \sin 2\mu \cdot d\mu - \frac{3}{2}\lambda(\cos 2\mu - \cos 2M)d\varepsilon,$$

$$\therefore d\varepsilon = \frac{d\lambda}{\lambda} \frac{2}{3(\cos 2\mu - \cos 2M)},$$

neglecting extremely small quantities of the higher order.

Now in the case before us,

$$\lambda = \text{latitude of Kalia} - \text{latitude of Kalianpur},$$

$$= 29^\circ 30' 48'' - 24^\circ 7' 11'' = 5^\circ 23' 37'',$$

$$\begin{aligned}\mu &= \text{half the sum of these latitudes,} \\ &= 26^\circ 49'; \cos 2\mu = 0.59295.\end{aligned}$$

Suppose  $d\lambda = 1''$  only; then

$$\begin{aligned}d\varepsilon &= \frac{1''}{5^\circ 23' 38''} \frac{2}{3(0.59295 - \cos 2M)} \\ &= \frac{1}{58254} \frac{2}{0.59295 - \cos 2M}\end{aligned}$$

This will be smallest when  $2M$  is chosen as nearly  $180^\circ$  as possible. The great arc lately measured near North Cape is the one which will best meet this condition. Put therefore  $M = 70^\circ$ ,  $\cos 2M = -0.76604$ , and

$$\begin{aligned}\therefore d\varepsilon &= \frac{1}{58254} \frac{2}{1.35899} = \frac{1}{39585} \\ &= \frac{\varepsilon}{132}, \text{ if we put } \frac{1}{300} \text{ for } \varepsilon.\end{aligned}$$

Hence for an error of  $5'' \cdot 236$  in defect in the amplitude, the effect on the ellipticity will be to diminish it by  $\frac{5 \cdot 236}{132} \varepsilon = \frac{\varepsilon}{25}$  nearly, or by nearly  $\frac{1}{25}$ th part of its whole value, under the most favourable circumstances. This is sufficient to show the great importance of endeavouring to account satisfactorily for the discrepancy brought to light by the Indian Survey; and that, not by merely putting it down to mountain attraction, but by calculating that attraction by some independent means, with a view to see whether its amount actually corresponds with the observed anomaly\*.

10. To dissect and actually to calculate the attraction of the masses of which the Himalayas, and the regions beyond, are composed, appears, at the very thought of it, to be an herculean undertaking next to impossible. I am fully convinced, however, that no other method will succeed. It is upon this plan that the solution of the problem is conducted in this paper. It will be seen, that by selecting a peculiar law of dissection the calculation is very greatly simplified, and made to depend entirely and solely upon a knowledge of the elevations and depressions, in fact, the general contour of the surface. This information for some part of the mass is already supplied by the maps of the Trigonometrical Survey.

11. In the following pages I propose, in the first place, to develop my method of calculation, and to deduce a formula by which the attraction can be determined with a precision corresponding to the degree of accuracy to which the contour of the surface is known.

\* If the effect of mountain attraction upon the northern division of the arc be what I make it,  $15'' \cdot 885$ , then the ellipticity as determined from this and the Russian arc would be too small by  $\frac{1}{8} \varepsilon$ , if mountain attraction is neglected. The error in the ellipticity in comparing the whole arc between Kaliana and Damargida with the North Cape arc will, under the same circumstances, amount even to  $\frac{1}{6} \varepsilon$ . This will appear from the sequel, and is here mentioned only to illustrate the importance of the subject under consideration.

In the second place, I propose to reduce the formula to numbers, and so arrive at such an approximate value of the attraction as the data I have been able to collect will allow.

12. This approximate value is, as will be seen, larger than  $5''\cdot236$ , the error brought to light by the Survey. I make various suppositions with a view, if possible, to reduce my result to this, but without effect. This leads me to look in another direction for an explanation of the cause of discordance, and I arrive at a conclusion which clears up the discrepancy, confirms the calculations of this paper, and illustrates the importance of not disregarding the influence of mountain attraction.

I. *Determination of a Formula for calculating Mountain Attraction on the stations of the Indian Arc.*

13. Let O be the centre of a circle AQ, AT the tangent at A, QR a slender prism of mass M, being the prolongation of the radius through the point Q. Then if  $AQ=a$ ,  $AR=b$ ,  $\angle QAR=\omega$ , and  $\angle AOQ=\theta$ , the following is true:—

*Lemma.*—The attraction of the prism QR on the point A in the direction AT

$$= \frac{M}{ab} \cos \frac{\theta}{2} \left\{ 1 + \tan \frac{\theta}{2} \tan \frac{\omega}{2} \right\}.$$

For let P be any point of the prism,  $QP=z$ ,  $QR=h$ ,  $\angle PAQ=\psi$ ,

$$\therefore \text{mass of element of prism at P} = M \frac{dz}{h},$$

$$\text{attraction of this on A in direction AP} = M \frac{dz}{h} \frac{1}{PA^2},$$

$$\text{attraction of this on A in direction AT} = M \frac{dz}{h} \frac{\cos PAT}{PA^2},$$

$$\cos PAT = \cos \left( \frac{1}{2}\theta - \psi \right)$$

$$\frac{AP}{a} = \frac{\cos \frac{1}{2}\theta}{\cos \left( \frac{1}{2}\theta + \psi \right)}, \quad \frac{h}{b} = \frac{\sin \omega}{\cos \frac{1}{2}\theta},$$

$$z = QP = a \frac{\sin \psi}{\cos \left( \frac{1}{2}\theta + \psi \right)} = a \left( \cos \frac{1}{2}\theta \tan \left( \frac{1}{2}\theta + \psi \right) - \sin \frac{1}{2}\theta \right)$$

$$\frac{dz}{d\psi} = a \cos \frac{1}{2}\theta \sec^2 \left( \frac{1}{2}\theta + \psi \right).$$

Fig. 2.

